

DEGREE 5 INVARIANT OF E_8

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1. INTRODUCTION

Throughout this paper, k denotes a field of characteristic 0. We write E_8 for the split simple algebraic group with Killing-Cartan type E_8 . The Galois cohomology set $H^1(k, E_8)$ classifies simple algebraic groups of type E_8 over k . One of the goals of the theory of algebraic groups over arbitrary fields is to understand the groups of type E_8 , equivalently, understand the set $H^1(k, E_8)$.

The main tool is the *Rost invariant*

$$r_{E_8}: H^1(*, E_8) \rightarrow H^1(*, \mu_{60}^{\otimes 2})$$

discovered by Markus Rost and explained in Merkurjev's portion of the book [GMS]. It is a morphism of functors from the category of fields over k to the category of pointed sets. We put

$$H^1(k, E_8)_0 := \{\eta \in H^1(k, E_8) \mid r_{E_8}(\eta) \text{ has odd order}\}.$$

In [Sem08, Corollary 8.7], the second author proved that there is a morphism of functors:

$$u: H^1(*, E_8)_0 \rightarrow H^5(*, \mathbb{Z}/2\mathbb{Z}).$$

This is the degree 5 invariant from the title.

Let now G be a group of type E_8 . It corresponds with a canonical element of $H^1(k, E_8)$, so it makes sense to speak of “the Rost invariant of G ”; we denote it by $r(G)$. Suppose now that $r(G)$ has odd order, so G belongs to $H^1(k, E_8)_0$. The second author also proved in [Sem08]:

$$(1.1) \quad \begin{array}{l} u(G) = 0 \text{ if and only if there is an odd-degree extension} \\ \text{of } k \text{ that splits } G. \end{array}$$

For example, the compact group G of type E_8 over \mathbb{R} has Rost invariant zero and $u(G) = (-1)^5$.

As an obvious corollary, we have:

$$(1.2) \quad \begin{array}{l} \text{If } k \text{ has cohomological dimension } \leq 2, \text{ then every } k\text{-group} \\ \text{of type } E_8 \text{ is split by an odd-degree extension of } k. \end{array}$$

Serre's "Conjecture II" for groups of type E_8 is that in fact every group of type E_8 over such a field is split.

The purpose of this note is to calculate the value of $u(G)$ for certain $G \in H^1(k, E_8)_0$ and to give some applications of u .

2. TITS'S CONSTRUCTION OF GROUPS OF TYPE E_8

2.1. There are inclusions of algebraic groups $G_2 \times F_4 \subset E_8$, where G_2 and F_4 denote split groups of those types. Furthermore, this embedding is essentially unique. Applying Galois cohomology gives a function $H^1(k, G_2) \times H^1(k, F_4) \rightarrow H^1(k, E_8)$. The first two sets classify octonion k -algebras and Albert k -algebras respectively, so this map gives a construction by Galois descent of groups of type E_8 :

$$\boxed{\text{octonion } k\text{-algebras}} \times \boxed{\text{Albert } k\text{-algebras}} \rightarrow \boxed{\text{groups of type } E_8}$$

Jacques Tits gave concrete formulas on the level of Lie algebras for this construction in [T], see also [J]. This method of constructing groups of type E_8 is known as the *Tits construction*. (Really, Tits's construction is more general and gives other kinds of groups as well. The variety of possibilities is summarized in Freudenthal's magic square as in [Inv, p. 540]. However, the flavor in all cases is the same, and this case is the most interesting.)

Our purpose is to compute the value of u on those groups of type E_8 with Rost invariant of odd order (so that it makes sense to speak of u) and arising from Tits's construction. We do this in Proposition 2.6.

2.2. Following [Inv], we write $f_3(-)$ for the even component of the Rost invariant of an Albert algebra or an octonion algebra (equivalently, a group of type F_4 or G_2). We write $g_3(-)$ for the odd component of the Rost invariant of an Albert algebra; such algebras also have an invariant f_5 taking values in $H^5(k, \mathbb{Z}/2\mathbb{Z})$. *An Albert algebra A has $g_3(A) = 0$ and $f_5(A) = 0$ iff A has a nonzero nilpotent, iff the group $\text{Aut}(A)$ is isotropic.*

Suppose now that $G \in H^1(k, E_8)$ is the image of an octonion algebra O and an Albert algebra A . It follows from a twisting argument as in the proof of Lemma 5.8 in [GQ] that

$$r(G) = r_{G_2}(O) + r_{F_4}(A).$$

In particular, G belongs to $H^1(k, E_8)_0$ if and only if $f_3(O) + f_3(A) = 0$ in $H^3(k, \mathbb{Z}/2\mathbb{Z})$, i.e., if and only if $f_3(O) = f_3(A)$.

2.3. Definition. Define

$$t: H^1(*, F_4) \rightarrow H^1(*, E_8)_0$$

by sending an Albert k -algebra A to the group of type E_8 constructed from A and the octonion algebra with norm form $f_3(A)$, via Tits's construction from 2.1. By the preceding paragraph, $r(G) = g_3(A) \in H^3(k, \mathbb{Z}/3\mathbb{Z})$, so G does indeed belong to $H^1(k, E_8)_0$.

2.4. Example. *If A has a (nonzero) nilpotent element, then the group $t(A)$ is split.* Indeed, $g_3(A)$ is zero so $t(A)$ is in the kernel of the Rost invariant. Also, $t(A)$ is isotropic because it contains the isotropic subgroup $\text{Aut}(A)$, hence $t(A)$ is split by, e.g., [Ga, Prop. 12.1(1)].

2.5. Example. In case $k = \mathbb{Q}$, there are exactly three Albert algebras up to isomorphism. All have $g_3 = 0$; they are distinguished by the values of f_3 and f_5 . :

$f_3(A)$	$f_5(A)$	$t(A)$
0	0	split by Example 2.4
$(-1)^3$	0	split by Example 2.4
$(-1)^3$	$(-1)^5$	anisotropic by [J, p. 118]

It follows from Chernousov's Hasse Principle for groups of type E_8 [PR] that for every number field K with a unique real place, the set $H^1(K, E_8)_0$ has two elements: the split group and the anisotropic group constructed as in the last line of the table.

2.6. Proposition. *For every Albert k -algebra A , we have:*

$$u(t(A)) = f_5(A) \in H^5(k, \mathbb{Z}/2\mathbb{Z}).$$

Proof. The composition ut is an invariant $H^1(*, F_4) \rightarrow H^5(*, \mathbb{Z}/2\mathbb{Z})$, hence is given by

$$ut(A) = \lambda_5 + \lambda_2 \cdot f_3(A) + \lambda_0 \cdot f_5(A)$$

for uniquely determined elements $\lambda_i \in H^i(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$, see [GMS, p. 50].

We apply this formula to each of the three lines in the table from Example 2.5. Obviously u of the split E_8 is zero, so the first line gives:

$$0 = u(\text{split } E_8) = \lambda_5 \in H^5(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}).$$

Applying this to the second line gives:

$$0 = u(\text{split } E_8) = \lambda_2 \cdot (-1)^3 \in H^5(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}).$$

For the last line, u of the compact E_8 is $(-1)^5$ by (1.1), see the end of [Sem08] for details. We find:

$$(-1)^5 = u(\text{compact } E_8) = \lambda_0 \cdot (-1)^5,$$

so λ_0 equals 1 in $H^0(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

To show that $\lambda_2 = 0$ we proceed as follows. Consider the pure transcendental extension $F = \mathbb{Q}(x, y, z, a, b)$ and let H be the group of

type F_4 with $f_3(H) = (x, y, z)$, $f_5(H) = f_3(H) \cdot (a, b)$ and $g_3(H) = 0$. Then $ut(H) = f_5(H) + f_3(H) \cdot \lambda_2$.

Let K be a generic splitting field for the symbol $f_5(H)$. Since H_K is isotropic, the resulting group $t(H)$ of type E_8 is isotropic over K , and, since it has trivial Rost invariant, it splits over K [Ga, Prop. 12.1]. Obviously, $ut(H)$ is killed by K . Therefore $f_3(H) \cdot \lambda_2$ is zero over K . If $f_3(H) \cdot \lambda_2$ is zero over F , then by taking residues we see that λ_2 is zero in $H^2(\mathbb{Q}(a, b), \mathbb{Z}/2\mathbb{Z})$, hence also in $H^2(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})$. Otherwise, $f_3(H) \cdot \lambda_2$ is equal to $f_5(H)$ by [OVVo, Theorem 2.1], and again completing and taking residues with respect to the x -, y -, and z -adic valuations, we find that $\lambda_2 = (a, b) \in H^2(\mathbb{Q}(a, b), \mathbb{Z}/2\mathbb{Z})$. But this is impossible because λ_2 is defined over \mathbb{Q} . This proves that $\lambda_2 = 0$. \square

2.7. Corollary. *Let G be as in Proposition 2.6. Let Kill_- denote the Killing form of $-$ and E_8 the split group. Then*

$$\langle 60 \rangle (\text{Kill}_G - \text{Kill}_{E_8}) = 2^3 \cdot u(G) \in I^8(k).$$

Proof. Follows from [Ga, 13.5 and Example 15.9] and Proposition 2.6. \square

2.8. Example. Whatever field k of characteristic zero one starts with, there is an extension K/k that supports an anisotropic 5-Pfister quadratic form q_5 —one can adjoin 5 indeterminates to k , for example. Let q_3 be a 3-Pfister form dividing q_5 and let A be the Albert K -algebra with $f_d(A) = e_d(q_d)$ for $d = 3, 5$. The group $G := t(A)$ of type E_8 over K has Rost invariant zero yet $u(G) = f_5(A)$ nonzero by Proposition 2.6. In particular, G is not split, hence is anisotropic by [Ga, Prop. 12.1(1)].

Example 15.9 in [Ga] produced anisotropic groups of type E_8 in a similar manner, but used the Killing form to see that the resulting groups were anisotropic; that method does not work if -1 is a square in k . Roughly, Example 2.8 above exhibits more anisotropic groups because u is a finer invariant than the Killing form.

3. INVARIANTS OF $H^1(*, \text{Spin}_{16})_0$

Recall from [Inv, pp. 436, 437] that the Rost invariant of a class $\eta \in H^1(k, \text{Spin}_{16})$ is given by the formula

$$r_{\text{Spin}_{16}}(\eta) = e_3(q_\eta) \in H^3(k, \mathbb{Z}/2\mathbb{Z})$$

where q_η is the 16-dimensional quadratic form in $I^3 k$ corresponding to the image of η in $H^1(k, \text{SO}_{16})$ and e_3 is the Arason invariant. it follows that η belongs to the kernel of the Rost invariant if and only if q_η belongs to $I^4 k$.

We can quickly find some invariants of the kernel $H^1(k, \text{Spin}_{16})_0$ of the Rost invariant. For η in that set, q_η is $\langle \alpha_\eta \rangle \gamma$ for some $\alpha_\eta \in k^\times$ and some 4-Pfister quadratic form γ [Lam, X.5.6]. (One can take α_η to be any element of k^\times represented by q_η [Lam, X.1.8].) We define invariants $f_d: H^1(*, \text{Spin}_{16})_0 \rightarrow H^d(*, \mathbb{Z}/2\mathbb{Z})$ for $d = 4, 5$ via:

$$f_4(\eta) := e_4(q_\eta) \quad \text{and} \quad f_5(\eta) := (\alpha_\eta) \cdot e_4(q_\eta),$$

where e_4 is the usual additive map $I^4(*) \rightarrow H^4(*, \mathbb{Z}/2\mathbb{Z})$. If q_η is isotropic, then q_η is hyperbolic and $e_4(q_\eta)$ is zero, so the value of $f_5(\eta)$ depends only on η and not on the choice of α_η , see [Ga 09, 10.2].

We can identify two more (candidates for) invariants of $H^1(*, \text{Spin}_{16})_0$. The split E_8 has a subgroup isomorphic to HSpin_{16} , the nontrivial quotient of Spin_{16} that is neither adjoint (i.e., not PSO_{16}) nor SO_{16} . Further, the composition

$$H^1(*, \text{Spin}_{16}) \rightarrow H^1(*, \text{HSpin}_{16}) \rightarrow H^1(*, E_8) \xrightarrow{r_{E_8}} H^3(*, \mathbb{Z}/60\mathbb{Z}(2))$$

is the Rost invariant of Spin_{16} [Ga, (5.2)]. We find a morphism of functors

$$H^1(*, \text{Spin}_{16})_0 \rightarrow H^1(*, E_8)_0.$$

Composing this with the invariant u gives an invariant

$$u_5: H^1(*, \text{Spin}_{16})_0 \rightarrow H^5(*, \mathbb{Z}/2\mathbb{Z}).$$

(Roughly speaking, we have used the invariant u of $H^1(*, E_8)_0$ to get an invariant of $H^1(*, \text{Spin}_{16})_0$ in the same way that Rost used the f_5 invariant of $H^1(*, F_4)$ to get an invariant of $H^1(*, \text{Spin}_9)$, see [Rost] or [Ga 09, 18.9].)

The purpose of this section is to prove:

3.1. Proposition. *The invariants $H^1(*, \text{Spin}_{16})_0 \rightarrow H^\bullet(*, \mathbb{Z}/2\mathbb{Z})$ form a free $H^\bullet(k, \mathbb{Z}/2\mathbb{Z})$ -module with basis*

$$1, f_4, f_5, u_5, u_6,$$

where the invariant u_6 is given by the formula $u_6(\eta) := (\alpha_\eta) \cdot u_5(\eta)$.

We first replace Spin_{16} with a more tractable group. The first author described in [Ga, §11] a subgroup of HSpin_{16} isomorphic to $\text{PGL}_2^{\times 4}$. Examining the root system data for this subgroup given in Tables 7B and 11 of that paper, we see that the inverse image of this subgroup in Spin_{16} is a subgroup H obtained by modding $\text{SL}_2^{\times 4}$ out by the subgroup generated by $(-1, -1, 1, 1)$, $(-1, 1, -1, 1)$, and $(-1, 1, 1, -1)$. Indeed, in the notation of that paper, each copy of $\text{PGL}_2^{\times 4}$ lifts to a copy of

SL_2 with a maximal torus the image of one of the four elements of the root lattice of D_8 :

$$\begin{aligned} \delta_1 + 2\delta_2 + 3\delta_3 + 4\delta_4 + 3\delta_5 + 2\delta_6 + \delta_7, \quad \delta_1 - \delta_3 + \delta_5 - \delta_7, \\ \delta_1 + \delta_3 - \delta_5 - \delta_7, \quad \text{or} \quad \delta_1 + 2\delta_2 + \delta_3 - \delta_5 - 2\delta_6 - \delta_7. \end{aligned}$$

(Here δ_i denotes the simple root of D_8 that is denoted α_i in [B].) The center of each copy of SL_2 has nontrivial element

$$h_{\delta_1}(-1) h_{\delta_3}(-1) h_{\delta_5}(-1) h_{\delta_7}(-1)$$

in the notation of [St]. This defines a homomorphism $\mu_2 \rightarrow H$ that gives a short exact sequence:

$$1 \rightarrow \mu_2 \rightarrow H \rightarrow \mathrm{PGL}_2^{\times 4} \rightarrow 1.$$

The image of $H^1(k, H)$ in $H^1(k, \mathrm{PGL}_2^{\times 4})$ consists of quadruples (Q_1, Q_2, Q_3, Q_4) of quaternion algebras so that $Q_1 \otimes Q_2 \otimes Q_3 \otimes Q_4$ is split.

Let φ map the Klein four-group $V := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ into $(\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2$ via

$$\varphi(1, 0) := ((\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})) \quad \text{and} \quad \varphi(0, 1) := ((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})).$$

This defines a homomorphism. Twisting $(\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2$ by a pair $(a, b) \in k^\times/k^{\times 2} \times k^\times/k^{\times 2} = H^1(k, V)$ gives $(\mathrm{SL}(Q) \times \mathrm{SL}(Q))/\mu_2$, where Q denotes the quaternion algebra (a, b) . (Of course, composing φ with either of the projections $(\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2 \rightarrow \mathrm{PGL}_2$ sends (a, b) to the same quaternion algebra Q .) The composition

$$V \times V \xrightarrow{\varphi \times \varphi} (\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2 \times (\mathrm{SL}_2 \times \mathrm{SL}_2)/\mu_2 \rightarrow H$$

gives a map whose image does not meet the center of Spin_{16} , which we denote by Z . This gives a homomorphism $Z \times V \times V \rightarrow \mathrm{Spin}_{16}$.

3.2. Lemma. *For every extension K/k , the map $Z \times V \times V \rightarrow \mathrm{Spin}_{16}$ defined above induces a map $H^1(K, Z \times V \times V) \rightarrow H^1(K, \mathrm{Spin}_{16})_0$ that is a surjection.*

Proof. Fix $\nu \in H^1(K, Z \times V \times V)$; we write ζ for its image in $H^1(K, Z)$ and Q_1, Q_2 respectively for its images under the two projections $H^1(K, Z \times V \times V) \rightarrow H^1(K, V) \rightarrow H^1(K, \mathrm{PGL}_2)$. It follows from the description of the subgroup $\mathrm{PGL}_2^{\times 4}$ in HSpin_{16} and the map $H^1(K, \mathrm{HSpin}_{16}) \rightarrow H^1(K, \mathrm{PSO}_{16})$ in [Ga, §4] that the image of ν in $H^1(K, \mathrm{PSO}_{16})$ is the same as the image of the quadratic form $q_1 \otimes q_2$ under $H^1(K, \mathrm{SO}_{16}) \rightarrow H^1(K, \mathrm{PSO}_{16})$, where q_i denotes the norm form of Q_i . As the Rost invariant of Spin_{16} “factors through” SO_{16} to give the Arason invariant, the image of ν in $H^1(K, \mathrm{Spin}_{16})$ is in the kernel of the Rost invariant.

As for surjectivity, if η is in $H^1(K, \mathrm{Spin}_{16})_0$ then $q_\eta = \langle \alpha \rangle q_1 q_2$ where q_1, q_2 are 2-Pfister forms. We define ν to be $(0, Q_1, Q_2)$ where Q_i is a

quaternion algebra with norm q_i . Then the image of ν in $H^1(K, \text{Spin}_{16})$ is $\zeta \cdot \eta$ for some $\zeta \in H^1(K, Z)$, and we deduce that $\zeta \cdot \nu$ maps to η . \square

It follows from the lemma (using [Ga09, 5.3]) that the module of invariants $H^1(*, \text{Spin}_{16})_0 \rightarrow H^\bullet(*, \mathbb{Z}/2\mathbb{Z})$ injects into the module of invariants $H^1(*, Z \times V \times V) \rightarrow H^\bullet(*, \mathbb{Z}/2\mathbb{Z})$. But we know this larger module by [GMS, p. 40] or [Ga09, 6.7]: it is spanned by products $\pi_1 \pi_2 \pi_3$ for $\pi_1 : H^1(*, Z) \rightarrow H^\bullet(*, \mathbb{Z}/2\mathbb{Z})$ and π_2, π_3 invariants $H^1(*, V) \rightarrow H^\bullet(*, \mathbb{Z}/2\mathbb{Z})$ composed with projection on the 2nd or 3rd term in the product.

For the rest of this section, we use the notation (ζ, Q_1, Q_2) for elements of $H^1(K, Z \times V \times V)$ as in the proof of Lemma 3.2.

3.3. Lemma. *Fix $(\zeta, Q_1, Q_2) \in H^1(K, Z \times V \times V)$. If $[Q_1] \cdot [Q_2] = 0$ in $H^4(K, \mathbb{Z}/2\mathbb{Z})$, then the image of (ζ, Q_1, Q_2) in $H^1(K, \text{Spin}_{16})$ is zero.*

Proof. The image η of (ζ, Q_1, Q_2) has $q_\eta = \langle \alpha_\eta \rangle q_1 q_2$ where q_i is the norm of Q_i . If $[Q_1] \cdot [Q_2]$ is zero, then $q_1 \otimes q_2$ is in $I^5(k)$ and so is hyperbolic. It follows that the image of η in $H^1(K, \text{PSO}_{16})$ is zero, hence η is in the image of $H^1(K, Z)$. But Spin_{16} is split semisimple, so the image of H^1 of the center in H^1 of the group is zero. \square

Proof of Proposition 3.1. The invariant f_4 sends (ζ, Q_1, Q_2) to $[Q_1] \cdot [Q_2]$. Lemma 3.3 combined with arguments like those in [GMS, pp. 43, 44] shows that every invariant of $H^1(*, \text{Spin}_{16})_0$ restricts to one of the form $\lambda + \phi \cdot f_4$ for uniquely determined $\lambda \in H^\bullet(k, \mathbb{Z}/2\mathbb{Z})$ and $\phi : H^1(*, Z) \rightarrow H^\bullet(*, \mathbb{Z}/2\mathbb{Z})$, i.e., is given by the formula

$$(\zeta, Q_1, Q_2) \mapsto \lambda + \phi(\zeta) \cdot [Q_1] \cdot [Q_2].$$

The collection of such invariants of $H^1(*, Z \times V \times V)$ forms a free $H^\bullet(k, \mathbb{Z}/2\mathbb{Z})$ -module with basis

$$1, \quad f_4, \quad \chi_v \cdot f_4, \quad \chi_h \cdot f_4, \quad \chi_v \cdot \chi_h \cdot f_4,$$

where χ_v and χ_h denote the maps $H^1(*, Z) \rightarrow H^1(*, \mathbb{Z}/2\mathbb{Z})$ defined by restricting to Z the vector representation $\text{Spin}_{16} \rightarrow \text{SO}_{16}$ and the half-spin representation $\text{Spin}_{16} \rightarrow \text{HSpin}_{16}$ implicit in our root-system description above. Obviously, f_5 restricts to be $\chi_v \cdot f_4$.

At this point, it suffices to prove:

$$(3.4) \quad u_5 \text{ restricts to be } \chi_h \cdot f_4.$$

Indeed, this statement implies that u_5 is zero when q_η is isotropic, hence by [Ga09, 10.2] the formula for u_6 gives a well-defined invariant; it obviously restricts to $\chi_v \cdot u_5 = \chi_v \cdot \chi_h \cdot f_4$ on $H^1(*, Z \times V \times V)$. Spanning and linear independence follow from the previous paragraph.

We now prove (3.4). The restriction of u_5 sends zero to zero, so it is $\phi \cdot f_4$ for some $\phi: H^1(*, Z) \rightarrow H^1(*, \mathbb{Z}/2\mathbb{Z})$ that itself sends zero to zero. Therefore (by [GMS, p. 40]) ϕ is induced by some homomorphism $\chi: Z \rightarrow \mathbb{Z}/2\mathbb{Z}$. As u_5 is defined by pulling back along the map $\text{Spin}_{16} \rightarrow E_8$, one quickly sees that χ must be zero or χ_h . As χ , the zero invariant, and χ_h are all defined over \mathbb{Q} , it suffices to prove that χ is not the zero invariant in the case where $k = \mathbb{Q}$. Example 15.1 of [Ga] gives a class $\nu \in H^1(\mathbb{R}, Z \times V \times V)$ whose image in $H^1(\mathbb{R}, E_8)$ is the compact E_8 , on which u is nonzero. Hence the restriction of u_5 to $Z \times V \times V$ is not zero over \mathbb{Q} and must be $\chi_h \cdot f_4$. \square

4. ESSENTIAL DIMENSION OF $H^1(*, \text{Spin}_{16})_0$

The following is a corollary of the previous section:

4.1. Corollary. *The essential dimension of the functor $H^1(*, \text{Spin}_{16})_0$ over every field of characteristic zero is 6.*

Proof. The existence of the nonzero invariant $u_6: H^1(*, \text{Spin}_{16})_0 \rightarrow H^6(*, \mathbb{Z}/2\mathbb{Z})$ implies that the essential dimension is at least 6 by [RY, Lemma 6.9]; this is the interesting inequality. One can deduce that the essential dimension is at most 6 by, for example, the surjectivity in Lemma 3.2. \square

By way of contrast, the essential dimension of the functor $H^1(*, \text{Spin}_{16})$ (without restricting to the kernel of the Rost invariant) is 24 by [BRV, Remark 3.9].

5. GALOIS DESCENT FOR REPRESENTATIONS OF FINITE GROUPS

In this section, we restate some observations of Serre from [Serre00] and [GR98] regarding projective embeddings of simple groups in exceptional algebraic groups. Combining these results with the u -invariant for E_8 gives some new embeddings results, see Example 5.5 below.

Let A be an abstract finite group and G a split semisimple linear algebraic group defined over \mathbb{Q} . Fix a faithful representation $G \rightarrow \text{GL}_N$ defined over \mathbb{Q} . Further on fix a monomorphism $\varphi: A \rightarrow G(\bar{k})$, where \bar{k} is an algebraically closed field of characteristic zero.

5.1. Definition. Let F be a field extension of \mathbb{Q} . We say that the character of a representation $A \rightarrow G(\bar{k}) \rightarrow \text{GL}_N(\bar{k})$ is *defined over F* , if all its values belong to F .

Let χ be the character of the representation

$$\varphi: A \rightarrow G(\bar{k}) \rightarrow \text{GL}_N(\bar{k})$$

and F its field of definition. Assume additionally that $Z_{G(\bar{k})}(A) = 1$ and that there is exactly one $G(\bar{k})$ -conjugacy class of homomorphisms $A \rightarrow G(\bar{k})$ with character χ .

The following theorem can be extracted from Serre's paper [Serre 00, 2.5.3]:

5.2. Theorem. *In the above notation there exists a twisted form G_0 of G defined over F together with a monomorphism $A \rightarrow G_0(F)$. Moreover, for a field extension K/F there is a representation $A \rightarrow G(K)$ with character χ iff $G \simeq G_0$ over K .*

Proof. Let

$$P = \{\alpha: A \rightarrow G \mid \alpha \text{ is a representation with character } \chi\}.$$

Then G acts on P by conjugation. By assumptions on A and G this action is transitive. Moreover, the condition on the centralizer guarantees that this action is simply transitive, i.e., for any $\alpha, \beta \in P(\bar{K})$ there exists a unique $g \in G(\bar{K})$ with $\beta = \alpha^g$. Thus, P is a G -torsor. Since all values of the character χ belong to the field F , the torsor P is defined over F .

Let $\eta \in H^1(F, G)$ be the 1-cocycle corresponding to the torsor P . Then $\sigma \cdot \varphi = \eta_\sigma^{-1} \varphi \eta_\sigma$ for all σ in the absolute Galois group $\text{Gal}(\bar{F}/F)$. Define now G_0 as the twisted form of G over F by the torsor P . The group G_0 is defined out of $G(\bar{F})$ by a twisted Galois action:

$$\sigma * g = \eta_\sigma(\sigma \cdot g) \eta_\sigma^{-1}.$$

Now it is easy to see that the homomorphism $\varphi: A \rightarrow G(\bar{F})$ is an F -defined homomorphism $A \rightarrow G_0(F)$.

Let K/F be a field extension. If there is a representation $A \rightarrow G(K)$ with character χ , then obviously G and G_0 are isomorphic over K . Conversely, if G and G_0 are isomorphic over K , then the image of the cocycle η in $H^1(K, \text{Aut}(G))$ is zero. Since the centralizer of A in G is trivial, the group G is adjoint. Therefore η is already zero in $H^1(K, G)$. \square

To characterize the isomorphism criterion of Theorem 5.2 we need the following proposition.

5.3. Proposition. *For each Killing-Cartan type Φ in the table*

Type Φ	F_4	G_2	E_8
n	3	3	5

there is a unique algebraic group G_0 of type Φ that is compact at every real place of every (equivalently, a particular) number field; it is defined

over \mathbb{Q} . For every field K of characteristic zero and n as in the table, the following are equivalent:

- (1) $G_0 \otimes K$ is split.
- (2) $(-1)^n = 0 \in H^n(K, \mathbb{Z}/2)$.
- (3) -1 is a sum of 2^{n-1} squares of the field K .

Proof. The first sentence is a standard part of the Kneser-Harder-Chernousov Hasse principle. The group G_0 is split at every finite place.

For the second claim, all cases but E_8 are well-known. For E_8 , if $G_0 \otimes K$ is split, then $(-1)^5$ is zero by the existence of u ; see 1.1. For the converse, G_0 equals $t(A)$ where A is the unique Albert \mathbb{Q} -algebra with no nilpotents (see Example 2.5). If $(-1)^5 = 0$ — i.e., $f_5(A) = 0$ — is zero in $H^5(K, \mathbb{Z}/2\mathbb{Z})$, then $A \otimes K$ has nilpotents and $G_0 \otimes K$ is split by Example 2.4. \square

In the following examples we denote as Alt_l the alternating group of degree l and as $\zeta_l = e^{2\pi i/l}$ a primitive l -th root of unity.

5.4. Example (type G_2). Let G denote the split group of type G_2 , $A = G(\mathbb{F}_2)$ (resp. $\text{PSL}(2, 8)$, $\text{PSL}(2, 13)$), and K a field of characteristic zero. Then there is an embedding $A \rightarrow G(K)$ iff -1 is a sum of 4 squares of K and $\zeta_9 + \bar{\zeta}_9 \in K$ (for $\text{PSL}(2, 8)$), resp. $\sqrt{13} \in K$ (for $\text{PSL}(2, 13)$).

Indeed, fix the minimal fundamental representation $G \rightarrow \text{GL}_7$. By [A87, Theorem 9(3,4,5)] there is a representation $\varphi: A \rightarrow G(\bar{k})$ whose character χ is defined over $F = \mathbb{Q}$ (resp. $F = \mathbb{Q}(\zeta_9 + \bar{\zeta}_9)$, $F = \mathbb{Q}(\sqrt{13})$). Moreover, G acts transitively on the homomorphisms $A \rightarrow G(\bar{k})$ with character χ (see [A87] and [Griess, Cor. 1 and 2]).

By [A87, 9.3(1)] the representation φ is irreducible. Therefore $Z_{G(\bar{k})}(A) = 1$. Thus, all conditions of Theorem 5.2 are satisfied. Therefore there is a twisted form G_0 of G defined over F and an embedding $A \rightarrow G_0$.

In particular, there is an embedding $A \rightarrow G_0(\mathbb{R})$. Since any finite subgroup of a Lie group is contained in its maximal compact subgroup, it is easy to see that $G_0 \otimes_F \mathbb{R}$ is compact for all embeddings of F into \mathbb{R} . Moreover, by Theorem 5.2 we have an embedding $A \rightarrow G(K)$ iff G_0 and G are isomorphic over K . By Proposition 5.3 the latter occurs iff -1 is a sum of 4 squares of K .

(Thus, we have recapitulated the argument from [Serre00, 2.5.3]).

5.5. Example (type E_8). Let G denote the split group of type E_8 , $A = \text{PGL}(2, 31)$ (resp. $A = \text{SL}(2, 32)$), and K a field of characteristic zero. We view G as a subgroup of GL_{248} via the adjoint representation. There is an embedding $A \rightarrow G(K)$ iff -1 is a sum of 16 squares and $\zeta_{11} + \bar{\zeta}_{11} \in K$ (for $\text{SL}(2, 32)$).

Indeed, by [GR 98, Theorem 2.27 and Theorem 3.25] there exists an embedding $A \rightarrow G(\bar{k})$ whose character is defined over $F = \mathbb{Q}$ (resp. $F = \mathbb{Q}(\zeta_{11} + \bar{\zeta}_{11})$). Using [GR 98] one can check all conditions of Theorem 5.2 (cf. Example 5.4).

It follows by Theorem 5.2 that there is an embedding $A \rightarrow G_0(F)$ for some twisted form G_0 of G . Again as in Example 1 one can see that G_0 is the unique group such that $G_0 \otimes_F \mathbb{R}$ is compact for all embeddings of F into \mathbb{R} . Finally by Proposition 5.3 G and G_0 are isomorphic over a field extension K/F iff -1 is a sum of 16 squares in K .

Roughly speaking, we have added the facts about the compact E_8 contained in the proof of Proposition 5.3 (which uses the existence of the u -invariant) to Serre's appendix [GR 98, App. B].

In the same way one can get the following example:

5.6. Example (type A_1). Let $G = \mathrm{PGL}_2$, $A = \mathrm{Alt}_4$ (resp. Alt_5), and K a field of characteristic zero. Then there is an embedding $A \rightarrow G(K)$ iff -1 is a sum of 2 squares and for Alt_5 additionally $\sqrt{5} \in K$ (see [Serre 80]).

REFERENCES

- [A87] M. Aschbacher. *Chevalley groups of type G_2 as the group of a trilinear form*. J. Alg. **109** (1987), 193–259.
- [B] N. Bourbaki, *Lie groups and Lie algebras: Chapters 4–6*, Springer, 2002.
- [BRV] P. Brosnan, Z. Reichstein, and A. Vistoli, *Essential dimension, spinor groups, and quadratic forms*, to appear in Annals of Math.
- [Ga] S. Garibaldi. *Orthogonal involutions on algebras of degree 16 and the Killing form of E_8* , with an appendix by K. Zainoulline. To appear in Contemp. Math. Available from arxiv.org
- [Ga 09] ———, *Cohomological invariants: exceptional groups and Spin groups*, to appear in Memoirs of the Amer. Math. Soc., July 2009.
- [GMS] S. Garibaldi, A. Merkurjev, J-P. Serre. *Cohomological invariants in Galois cohomology*. University Lecture Series **28**, Amer. Math. Soc., Providence, RI, 2003.
- [GQ] S. Garibaldi and A. Quéguiner-Mathieu, *Restricting the Rost invariant to the center*. St. Petersburg Math. J. **19** (2008), no. 2, 197–213.
- [GR 98] R. Griess and A. Ryba, *Embeddings of $\mathrm{PGL}(2, 31)$ and $\mathrm{SL}(2, 32)$ in $E_8(\mathbb{C})$* . Appendices by M. Larsen and J-P. Serre. Duke Math. J. **94** (1998), 181–211.
- [Griess] R. Griess, *Basic conjugacy theorems for G_2* , Invent. math. **121** (1995), 257–277.
- [Inv] M.-A. Knus, A. Merkurjev, M. Rost, J.-P. Tignol. *The book of involutions*. AMS Colloquium Publ., vol. 44, 1998.
- [J] N. Jacobson, *Exceptional Lie algebras*. Lecture Notes in Pure and Applied Math., 1 Marcel Dekker, Inc., New York 1971.

- [Lam] T.-Y. Lam, *Introduction to quadratic forms over fields*, Amer. Math. Soc., Providence, RI, 2005.
- [OVVo] D. Orlov, A. Vishik, V. Voevodsky, *An exact sequence for $K_*^M/2$ with applications to quadratic forms*, Ann. of Math. **165** (2007), no. 1, 1–13.
- [PR] V.P. Platonov and A.S. Rapinchuk, *Algebraic groups and number theory*, Academic Press, Boston, 1994.
- [RY] Z. Reichstein and B. Youssin, *Essential dimension of algebraic groups and a resolution theorem for G -varieties*, Canad. J. Math. **52** (2000), 1018–1056.
- [Rost] M. Rost, *On the Galois cohomology of $\mathrm{Spin}(14)$* , Preprint 1999. Available from <http://www.math.uni-bielefeld.de/~rost>
- [Sem 08] N. Semenov. *Motivic construction of cohomological invariants*. Preprint 2008. Available from <http://www.math.uiuc.edu/K-theory/0909>
- [Serre 80] J-P. Serre *Extensions icosaédriques*. In *Seminar on Number Theory*, 1979–1980, exp. 19, Uni. Bordeaux I, Talence 1980.
- [Serre 00] ———, *Sous-groupes finis des groupes de Lie*. Astérisque **266** (2000), 415–430.
- [St] R. Steinberg, *Lectures on Chevalley groups*, Yale, 1968.
- [T] J. Tits, *Algèbres alternatives, algèbres de Jordan et algèbres de Lie exceptionnelles*. Indag. Math. **28** (1966), 223–237.

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